

TORIC POISSON IDEALS IN CLUSTER ALGEBRAS

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ABSTRACT. This paper investigates the Poisson geometry associated to a cluster algebra over the complex numbers, and its relationship to compatible torus actions. We show, under some assumptions, that each Noetherian cluster algebra has only finitely many torus invariant Poisson prime ideals and we show how to obtain using the exchange matrix of an initial seed. In fact, these ideals are independent of the choice of compatible Poisson structure. In many interesting cases the ideals can be described more explicitly. Cluster algebras and Poisson geometry

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1. INTRODUCTION

Cluster algebras were introduced by Fomin and Zelevinsky around the year 2000 ([9]) in order to understand the combinatorial properties of Lusztig's dual canonical

basis (see e.g. [33] and [34]) in quantum groups. Being commutative algebras, cluster algebras relate to these quantized function algebras via Poisson geometry—the *compatible Poisson structures* introduced by Gekhtman, Shapiro and Vainshtein in [15], whose properties are also the focus of their recent book [16]. Given a Noetherian Poisson algebra, it is a natural question to investigate the symplectic/Poisson geometry attached to it. As there exists a natural algebraic torus T which acts on the cluster algebra, we can follow the approach developed by Brown and Gordon in [4] and Goodearl in [19]. The first step, then, would be to classify the torus orbits of Poisson ideals of the algebra, for which we need to determine the torus invariant Poisson prime ideals, abbreviated as TPPs.

In the present paper we study TPPs in Noetherian (upper) cluster algebras using the combinatorial information obtained from the initial data, the seed of the cluster algebra. The main idea is that a cluster, and its nearby mutations should tell us much about the geometry attached to the cluster algebra as a whole.

Cluster algebras are nowadays very well-established, hence we do not recall any of the definitions here, and refer the reader to the literature, resp. our Section 2. We will denote the initial seed by (\mathbf{x}, B) where $\mathbf{x} = (x_1, \dots, x_n)$ and B is an integer $m \times n$ -matrix with $m \leq n$ such that its principal $m \times m$ submatrix is skew-symmetrizable. The cluster variables x_{m+1}, \dots, x_n are the frozen variables which we will call coefficients. The Poisson coefficient matrix Λ is a skew-symmetric $n \times n$ -matrix (see also Section 2.4). We refer to a cluster algebra with compatible Poisson bracket as a *Poisson cluster algebra*, given by (\mathbf{x}, B, Λ) .

Main Theorem 1.1. *Let \mathbb{A} be a Noetherian cluster algebra or upper cluster algebra over the complex numbers, given by (\mathbf{x}, B, Λ) , and T the torus of global toric actions. Assume that it is sufficiently generic (for details see Section 3). Then, there are only finitely many torus invariant Poisson prime ideals in \mathbb{A} .*

Moreover, we consider cluster algebras for which the following assumption called COS (see Condition 5.1 in Section 5) holds: Let \mathcal{I}, \mathcal{J} be TPPs, and let $\text{codim}(\mathcal{I}) = \text{codim}(\mathcal{J}) - k$. If $\mathcal{I} \subset \mathcal{J}$, then there exist TPPs $\mathcal{I} = \mathcal{I}_0, \mathcal{I}_1, \dots, \mathcal{I}_{k-1}, \mathcal{I}_k = \mathcal{J}$ such that $\mathcal{I} = \mathcal{I}_0 \subsetneq \mathcal{I}_1 \subsetneq \dots \subsetneq \mathcal{I}_{k-1} \subsetneq \mathcal{I}_k = \mathcal{J}$. Assuming COS, we can explicitly describe these ideals (Theorem 5.4), in terms of Laurent polynomials. It is known that COS holds for many interesting classes of cluster algebras, e.g. the algebra of functions on a complex semisimple algebraic group, introduced in [1], or unipotent radicals, studied in [12] and [13]. Moreover, we conjecture that all cluster algebras with a compatible Poisson structure satisfy COS (Conjecture 5.8). As evidence, we show that acyclic cluster algebras of even rank, which could not satisfy COS if they contained non-trivial TPPs, do, indeed, not have non-trivial TPPs (Theorem 4.1). We can use this result to prove that the corresponding cluster variety is smooth. The reader should notice that if \mathbb{A} is the coordinate ring of an affine variety X , then X does not necessarily equal the cluster manifold of [15], as X may not be smooth (see Example 3.30). The cluster manifold, however, is an open subset of X .

Let us briefly explain why this result may be interesting. The canonical, resp. dual canonical basis is not the only elusive feature about quantum groups. When ring theorists began to investigate the prime spectra of quantum groups and to relate them to their classical counterparts—symplectic leaves of the so called standard Poisson structure on semisimple complex algebraic groups—it became clear that

the following conjecture should hold (for notation and definitions see Goodearl's [19]):

Conjecture 1.2. *Let G be a complex semisimple algebraic group and $\mathcal{O}_q(G)$ the corresponding quantized function algebra. The topological space of primitive ideals, the primitive spectrum of $\mathcal{O}_q(G)$, is homeomorphic to the space of symplectic leaves of the standard Poisson structure on $\mathbb{C}[G]$, where the latter is endowed with the natural quotient topology.*

The conjecture is an analogue of Kirillov's Orbit Method, resp. Geometric Quantization (see [27] and [28]). In this context, such a homeomorphism is referred to as a Dixmier-map. Hodges, Levasseur [22], [23], Toro [24] and Joseph ([25],[26]) constructed a stratification of the prime and primitive spectra of quantum groups into torus orbits. Very recently, Yakimov formulated a precise version of Conjecture 1.2 in [54, Section 4], where he employed results of Kogan and Zelevinsky [29] in order to parametrize the symplectic leaves. Kogan and Zelevinsky's work foreshadows cluster algebras which appeared just a few years later. However, there is at present no way to study the topology of these spaces. Using the methods developed in the proof of Theorem 1.1 and Theorem 5.4, in particular by employing the concept of defining clusters, we hope to shed some light on the topology of the space of symplectic leaves. The recent progress regarding the theory of quantum cluster algebras (see e.g. Geiss, Leclerc and Schröer's [14]) leads us to believe that proving similar results for certain quantum cluster algebras will allow to establish continuity of the Dixmier map for example in the case of SL_n . Notice that we are not studying symplectic leaves in the present paper, but Poisson ideals which, in general, yield a coarser stratification—into symplectic cores rather than symplectic leaves (see e.g. [4]). However, we will explain in a future paper how we can apply our results to a study of symplectic leaves.

Let us briefly explain the organization of the paper, and some of the ideas of the proofs. First, we recall some definitions and well-known facts, about cluster algebras and their compatible Poisson structures. The subsequent Section 3 is devoted to the proof of the main theorem. First, we consider the intersection S of a toric Poisson prime ideal (TPP) \mathcal{I} with a certain finite set of cluster variables Y , and show that S must satisfy a number of conditions. The key observation is Proposition 3.10, which states that the intersection of \mathcal{I} with the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$ generated by a cluster $\mathbf{x} = (x_1, \dots, x_n)$ is generated by a subset of $\{x_1, \dots, x_n\}$. We next introduce the notion of a *defining cluster* for a TPP \mathcal{I} and construct such a cluster from a given cluster \mathbf{x} through mutations. The defining clusters allow us to prove existence and finiteness results which complete the proof of Theorem 1.1. As a non-trivial application we show in Section 4 that the cluster variety defined by an acyclic cluster algebra of even rank with trivial coefficients is smooth (always under the assumption that B has full rank).

In Section 5 we introduce our strongest result. Suppose the cluster algebra, or upper cluster algebra \mathbb{A} satisfies the condition COS (see above). We can now explicitly describe the TPPs and their inclusion relations (Corollary 5.7). Moreover, given an element $a \in \mathbb{A}$ and a TPP \mathcal{I} we can determine algorithmically whether $a \in \mathcal{I}$ (see Theorem 5.4). An appendix on torus invariant prime ideals completes the text. Additionally, we explain our constructions on a running example, the Grassmannian $\mathbb{C}[G(2, 5)]$ of two-dimensional subspaces of \mathbb{C}^5 .

Clearly, this paper is only a starting point. It suggests that if we manage to better understand the (Poisson) geometry associated with clusters and cluster algebras, we should be rewarded with important and beautiful results.

2. CLUSTER ALGEBRAS

2.1. Cluster algebras. In this section, we will review the definitions and some basic results on cluster algebras. Denote by $\mathfrak{F} = \mathbb{C}(x_1, \dots, x_n)$ the field of fractions in n indeterminates. Let B be a $m \times n$ -integer matrix such that its principal $m \times m$ -submatrix is skew-symmetrizable. Recall that a $m \times m$ -integer matrix B' is called skew-symmetrizable if there exists a $m \times m$ -diagonal matrix D with positive integer entries such that $B' \cdot D$ is skew-symmetric. We call the tuple (x_1, \dots, x_n, B) the *initial seed* of the cluster algebra and (x_1, \dots, x_m) a cluster, while $\mathbf{x} = (x_1, \dots, x_n)$ is called an extended cluster. The cluster variables x_{m+1}, \dots, x_n are called *coefficients*. We will now construct more clusters, (y_1, \dots, y_m) and extended clusters $\mathbf{y} = (y_1, \dots, y_n)$, which are transcendence bases of \mathfrak{F} , and the corresponding seeds (\mathbf{y}, \tilde{B}) in the following way.

Define for each real number r the numbers $r^+ = \max(r, 0)$ and $r^- = \min(r, 0)$. Given a skew-symmetrizable integer $m \times n$ -matrix B , we define for each $1 \leq i \leq m$ the *exchange polynomial*

$$(2.1) \quad P_i = \prod_{k=1}^n x_k^{b_{ik}^+} + \prod_{k=1}^n x_k^{-b_{ik}^-}.$$

We can now define the new cluster variable $x'_i \in \mathfrak{F}$ via the equation

$$(2.2) \quad x_i x'_i = P_i.$$

This allows us to refer to the matrix B as the *exchange matrix* of the cluster (x_1, \dots, x_n) , and to the relations defined by Equation 2.2 for $i = 1, \dots, m$ as *exchange relations*.

We obtain that $(x_1, x_2, \dots, \hat{x}_i, x'_i, x_{i+1}, \dots, x_n)$ is a transcendence basis of \mathfrak{F} . We now define the new exchange matrix $B_i = B' = (b'_{ij})$, associated to the new (extended) cluster

$$\mathbf{x}_i = (x_1, x_2, \dots, \hat{x}_i, x'_i, x_{i+1}, \dots, x_n)$$

by defining the coefficients b'_{ij} as follows:

- $b'_{ij} = -b_{ij}$ if $j \leq n$ and $i = k$ or $j = k$,
- $b'_{ij} = b_{ij} + \frac{|b_{ik}|b_{kj} + b_{ik}|b_{kj}|}{2}$ if $j \leq n$ and $i \neq k$ and $j \neq k$,
- $b'_{ij} = b_{ij}$ otherwise.

This algorithm is called *matrix mutation*. Note that B_i is again skew-symmetrizable (see e.g. [9]). The process of obtaining a new seed is called *cluster mutation*. The set of seeds obtained from a given seed (\mathbf{x}, B) is called the mutation equivalence class of (\mathbf{x}, B) .

Definition 2.1. The cluster algebra $\mathfrak{A} \subset \mathfrak{F}$ corresponding to an initial seed (x_1, \dots, x_n, B) is the subalgebra of \mathfrak{F} , generated by the elements of all the clusters in the mutation equivalence class of (\mathbf{x}, B) . We refer to the elements of the clusters as the cluster variables.

Remark 2.2. Notice that the coefficients, resp. frozen variables x_{m+1}, \dots, x_n will never be mutated. Of course, that explains their name.

We have the following fact, motivating the definition of cluster algebras in the study of total positivity phenomena and canonical bases.

Proposition 2.3. [9, Section 3] (*Laurent phenomenon*) *Let \mathfrak{A} be a cluster algebra with initial extended cluster (x_1, \dots, x_n) . Any cluster variable x can be expressed uniquely as a Laurent polynomial in the variables x_1, \dots, x_n with integer coefficients.*

Moreover, it has been conjectured for all cluster algebras, and proven in many cases (see e.g. [40] and [7],[8]) that the coefficients of these polynomials are positive.

Finally, we recall the definition of the lower bound of a cluster algebra \mathbb{A} corresponding to a seed (\mathbf{x}, B) . Denote by y_i for $1 \leq i \leq m$ the cluster variables obtained from \mathbf{x} through mutation at i ; i.e., they satisfy the relation $x_i y_i = P_i$.

Definition 2.4. [1, Definition 1.10] *Let \mathbb{A} be a cluster algebra and let (\mathbf{x}, B) be a seed. The lower bound $\mathfrak{L}_B \subset \mathbb{A}$ associated with (\mathbf{x}, B) is the algebra generated by the set $\{x_1, \dots, x_n, y_1, \dots, y_m\}$.*

2.2. Upper cluster algebras. Berenstein, Fomin and Zelevinsky introduced the related concept of upper cluster algebras in [1].

Definition 2.5. *Let $\mathfrak{A} \subset \mathfrak{F}$ be a cluster algebra with initial cluster (x_1, \dots, x_n, B) and let, as above, y_1, \dots, y_m be the cluster variables obtained by mutation in the directions $1, \dots, m$, respectively.*

(a) *The upper bound $\mathcal{U}_{\mathbf{x}, B}(\mathfrak{A})$ is defined as*

$$(2.3) \quad \mathcal{U}_{\mathbf{x}, B}(\mathfrak{A}) = \bigcap_{j=1}^m \mathbb{C}[x_1^{\pm 1}, \dots, x_{j-1}^{\pm 1}, x_j, y_j, x_{j+1}^{\pm 1}, \dots, x_m^{\pm 1}, x_{m+1}, \dots, x_n] .$$

(b) *The upper cluster algebra $\mathcal{U}(\mathfrak{A})$ is defined as*

$$\mathcal{U}(\mathfrak{A}) = \bigcap_{(\mathbf{x}', B')} \mathcal{U}_{\mathbf{x}'}(\mathfrak{A}) ,$$

where the intersection is over all seeds (\mathbf{x}', B') in the mutation equivalence class of (\mathbf{x}, B) .

Observe that each cluster algebra is contained in its upper cluster algebra (see [1]).

2.3. The Standard Example. We will now introduce our standard example—the coordinate ring $\mathbb{C}[G(2, 5)]$ of the Grassmannian $G(2, 5)$, which is the variety of two-dimensional subspaces of \mathbb{C}^5 . We define it as the subalgebra of the functions on 2×5 -matrices $\mathbb{C}[Mat_{2,5}]$, generated by the ten 2×2 -minors,

$$\Delta_{ij} = x_{1i}x_{2j} - x_{2i}x_{1j}$$

with $1 \leq i < j \leq 5$. It is well-known that the minors are subject to the Plücker relations

$$(2.4) \quad \Delta_{ik}\Delta_{j\ell} = \Delta_{ij}\Delta_{k\ell} + \Delta_{i\ell}\Delta_{jk} ,$$

for $1 \leq i < j < k < \ell \leq 5$. The algebra $\mathbb{C}[G(2, 5)]$ has a natural cluster algebra structure (see e.g. [49]). We can choose an initial seed with cluster variables $x_1 =$

Δ_{13} and $x_2 = \Delta_{14}$ and coefficients $x_3 = \Delta_{12}$, $x_4 = \Delta_{23}$, $x_5 = \Delta_{34}$, $x_6 = \Delta_{45}$, $x_7 = \Delta_{15}$. The corresponding exchange matrix is:

$$(2.5) \quad B = \begin{pmatrix} \mathbf{0} & \mathbf{1} & -1 & 1 & -1 & 0 & 0 \\ -1 & \mathbf{0} & 0 & 0 & 1 & -1 & 1 \end{pmatrix}.$$

The exchange relations are therefore:

$$(2.6) \quad \begin{aligned} \Delta_{13} y_1 &= \Delta_{14} \Delta_{23} + \Delta_{12} \Delta_{34}, \\ \Delta_{14} y_2 &= \Delta_{34} \Delta_{15} + \Delta_{13} \Delta_{45}. \end{aligned}$$

We observe from Equation 2.4 that $y_1 = \Delta_{24}$ and $y_2 = \Delta_{35}$. Indeed, the minors Δ_{ij} , with $1 \leq i < j \leq 5$, form the set of cluster variables. The cluster algebra is a cluster algebra of finite type A_2 in the classification of [10].

2.4. Poisson structures. Cluster algebras are closely related to Poisson algebras. In this section we recall some of the related notions and results.

Definition 2.6. *Let k be a field of characteristic 0. A Poisson algebra is a pair $(A, \{\cdot, \cdot\})$ of a commutative k -algebra A and a bilinear map $\{\cdot, \cdot\} : A \otimes A \rightarrow A$, satisfying for all $a, b, c \in A$:*

- (1) *skew-symmetry:* $\{a, b\} = -\{b, a\}$
- (2) *Jacobi identity:* $\{a, \{b, c\}\} + \{c, \{a, b\}\} + \{b, \{c, a\}\} = 0$,
- (3) *Leibniz rule:* $a\{b, c\} = \{a, b\}c + b\{a, c\}$.

If there is no room for confusion we will refer to a Poisson algebra $(A, \{\cdot, \cdot\})$ simply as A .

Gekhtman, Shapiro and Vainshtein showed in [15] that one can associate Poisson structures to cluster algebras in the following way. Let $\mathfrak{A} \subset \mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}] \subset \mathfrak{F}$ be a cluster algebra. A Poisson structure $\{\cdot, \cdot\}$ on $\mathbb{C}[x_1, \dots, x_n]$ is called log-canonical if $\{x_i, x_j\} = \lambda_{ij} x_i x_j$ with $\lambda_{ij} \in \mathbb{C}$ for all $1 \leq i, j \leq n$.

The Poisson structure can be naturally extended to \mathfrak{F} by using the identity $0 = \{f \cdot f^{-1}, g\}$ for all $f, g \in \mathbb{C}[x_1, \dots, x_n]$. We thus obtain that $\{f^{-1}, g\} = -f^{-2} \{f, g\}$ for all $f, g \in \mathfrak{F}$. We call $\Lambda = (\lambda_{ij})_{i,j=1}^n$ the *coefficient matrix* of the Poisson structure. We say that a Poisson structure on \mathfrak{F} is compatible with \mathfrak{A} if it is log-canonical with respect to each cluster (y_1, \dots, y_n) ; i.e., it is log canonical on $\mathbb{C}[y_1, \dots, y_n]$.

Remark 2.7. *A classification of Poisson structures compatible with cluster algebras was obtained by Gekhtman, Shapiro and Vainshtein in [15, Theorem 1.4].*

We will refer to the cluster algebra \mathbb{A} defined by the initial seed (\mathbf{x}, B) together with the compatible Poisson structure defined by the coefficient matrix Λ with respect to the cluster \mathbf{x} as the *Poisson cluster algebra* defined by the *Poisson seed* (\mathbf{x}, B, Λ) .

It is not obvious under which conditions a Poisson seed (\mathbf{x}, B, Λ) would yield a Poisson bracket $\{\cdot, \cdot\}_\Lambda$ on \mathfrak{F} such that $\{\mathbb{A}, \mathbb{A}\}_\Lambda \subset \mathbb{A}$. We have, however, the following fact.

Proposition 2.8. *Let (\mathbf{x}, B, Λ) be a Poisson seed and \mathbb{A} the corresponding cluster algebra. Then Λ defines a Poisson algebra structure on the upper bound $\mathcal{U}_{\mathbf{x}, B}(\mathbb{A})$ and the upper cluster algebra $\mathcal{U}(\mathbb{A})$.*

Proof. Denote as above by $\{\cdot, \cdot\}_\Lambda$ the Poisson bracket on \mathfrak{F} by Λ . Observe that the algebras $\mathbb{C}[x_1^{\pm 1}, \dots, x_{i-1}^{\pm 1}, x_i, y_i, x_{i+1}^{\pm 1}, \dots, x_n^{\pm 1}]$ are Poisson subalgebras of the Poisson algebra $\mathbb{C}[x_1^{\pm 1}, \dots, x_n^{\pm 1}]$ for each $1 \leq i \leq m$, as $\{x_i, y_i\}_\Lambda = \{x_i, x_i^{-1} P_i\}_\Lambda \in \mathbb{C}[x_1, \dots, x_n]$. If A is a Poisson algebra and $\{B_i \subset A : i \in I\}$ is a family of Poisson subalgebras, then $\bigcap_{i \in I} B_i$ is a Poisson algebra, as well. The assertion follows. \square

2.5. Toric Actions. We recall the definitions and properties of local and global toric actions from Gekhtman, Shapiro and Vainshtein [15] (see also [16]) where they are introduced in the context of cluster manifolds. As discussed in [15], the cluster manifold associated to a cluster algebra \mathbb{A} is not necessarily equal to the spectrum of maximal ideals of \mathbb{A} , even when \mathbb{A} is Noetherian. For example the corresponding variety may have singularities (see Example 3.30), and hence does not admit a manifold structure. The main notions, however, carry over into our context.

Let X be an affine variety such that $\mathfrak{A} = \mathbb{C}[X]$ is a cluster algebra or upper cluster algebra. Let $\mathbf{x} = (x_1, \dots, x_n)$ be a cluster. Following [15, Section 2.3] we define for each element $\mathbf{w} = (w_1, w_2, \dots, w_n) \in \mathbb{Z}^n$ a *local toric action* of \mathbb{C}^* on $\mathbb{C}[x_1, \dots, x_n]$ via maps $\psi_{\mathbf{x}, \alpha} : (x_1, \dots, x_n) \mapsto (\alpha^{w_1} x_1, \dots, \alpha^{w_n} x_n)$ for all $\alpha \in \mathbb{C}^*$. Assume now that we have chosen integer weights $\mathbf{w}_{\mathbf{x}} = (w_1, w_2, \dots, w_n)$ for each cluster \mathbf{x} . The local toric actions for two clusters are compatible if the following diagram commutes for any two clusters $\mathbf{x} = (x_1, \dots, x_n)$ and $\mathbf{y} = (y_1, \dots, y_n)$, connected by a sequence of mutations T :

$$\begin{array}{ccc} \mathbb{C}[\mathbf{x}] & \xrightarrow{T} & \mathbb{C}[\mathbf{y}] \\ \downarrow \psi_{\mathbf{x}, \alpha} & & \downarrow \psi_{\mathbf{y}, \alpha} \\ \mathbb{C}[\mathbf{x}] & \xrightarrow{T} & \mathbb{C}[\mathbf{y}] \end{array}.$$

Compatible local toric actions define a *global toric action* on the cluster algebra and a *toric flow* on X . We have the following fact.

Lemma 2.9. [15, Lemma 2.3] *Let B denote the exchange matrix of the cluster algebra at the cluster \mathbf{x} . The local toric action at \mathbf{x} defined by $\mathbf{w} \in \mathbb{Z}^n$ can be extended to a global toric action if and only if $B \cdot \mathbf{w} = 0$. Moreover, if such an extension exists, it is unique.*

We shall now discuss how to obtain all Poisson structures compatible with a cluster algebra \mathfrak{A} , given a Poisson seed (\mathbf{x}, B, Λ) where B is an $m \times n$ -matrix. Denote $k = n - m$. Let C be an integer $n \times k$ matrix. We define an action of the torus $(\mathbb{C}^*)^k$ on $\mathbb{C}[x_1, \dots, x_n]$ where $\mathbf{d} = (d_1, \dots, d_k) \in (\mathbb{C}^*)^k$ acts on x_i , $1 \leq i \leq n$, as

$$(2.7) \quad \mathbf{d} \cdot_C x_i = x_i \prod_{j=1}^m d_j^{c_{ij}}.$$

The local toric action extends to a global toric action of $(\mathbb{C}^*)^k$ on \mathbf{x} if and only if $B \cdot C = 0$ by Lemma 2.9. Notice that every skew-symmetric $k \times k$ -matrix V defines a Poisson bracket on $(\mathbb{C}^*)^k$ with $\{x_i, x_j\}_V = v_{ij} x_i x_j$. One obtains the following result.

Proposition 2.10. [17, Proposition 2.2] *Let $\mathcal{U}(\mathbb{A})$ be the Poisson upper cluster algebra defined by (\mathbf{x}, B, Λ) , and denote by $\{\cdot, \cdot\}_\Lambda$ the Poisson bracket. Let $\{\cdot, \cdot\}'$*

be another compatible Poisson structure and let $\{\cdot, \cdot\}'_\lambda$ be the bracket defined by $\{a, b\}'_\lambda = \lambda\{a, b\}'$. Then there exists a $n \times k$ -integer matrix C defining a global toric action, a skew-symmetric $k \times k$ matrix V and $\lambda \in \mathbb{C}$ such that the action of Equation 2.7 extends to a homomorphism of Poisson algebras

$$((\mathbb{C}^*)^m, \{\cdot, \cdot\}_V) \times (\mathcal{U}(\mathbb{A}), \{\cdot, \cdot\}_\Lambda) \longrightarrow (\mathcal{U}(\mathbb{A}), \{\cdot, \cdot\}'_\lambda) .$$

2.6. Toric Actions on Subalgebras. Let B be an exchange matrix as above and let $T = \ker(B)$. Let $\mathbf{i} = \{x_{i_1}, \dots, x_{i_k}\}$ be a k -element subset of \mathbf{x} and let for $\ell = n - k$ be $\{x_{j_1}, \dots, x_{j_\ell}\} = \mathbf{x} - \mathbf{i}$. Denote by $\mathbb{Z}^{\mathbf{i}}$ the sublattice of \mathbb{Z}^n spanned by e_{i_1}, \dots, e_{i_k} and by $T_{\mathbf{i}}$ the quotient $T/\mathbb{Z}^{\mathbf{i}}$. The global toric actions act on $\mathbb{C}[x_{j_1}, \dots, x_{j_\ell}]$ as follows: Let $t \in T$ and $\alpha \in \mathbb{C}^*$ then

$$t(\alpha)x_{j_h} = t_{\mathbf{i}}(\alpha)x_{j_h} ,$$

where $t_{\mathbf{i}}$ denotes the image of t under the natural projection of T onto $T_{\mathbf{i}}$. Notice that if B is generic, then $\text{rank}(T_{\mathbf{i}}) = \max(\text{rank}(T), n - |\mathbf{i}|)$.

2.7. Compatible Pairs and Their Mutation. Section 2.7 is dedicated to compatible pairs and their mutation. As we shall see below, compatible pairs yield important examples of Poisson brackets which are compatible with a given cluster algebra structure. Note that our definition is slightly different from the original one in [2]. Let, as above, $m \leq n$. Consider a pair consisting of a skew-symmetrizable $m \times n$ -integer matrix B with rows labeled by the interval $[1, m] = \{1, \dots, m\}$ and columns labeled by $[1, n]$ together with a skew-symmetrizable $n \times n$ -integer matrix Λ with rows and columns labeled by $[1, n]$.

Definition 2.11. *Let B and Λ be as above. We say that the pair (B, Λ) is compatible if the coefficients d_{ij} of the $m \times n$ -matrix $D = B \cdot \Lambda$ satisfy $d_{ij} = d_i \delta_{ij}$ for some positive integers d_i ($i \in [1, m]$).*

This means that $D = B \cdot \Lambda$ is a $m \times n$ matrix where the only non-zero entries are positive integers on the diagonal of the principal $m \times m$ -submatrix.

The following fact is obvious.

Lemma 2.12. *Let (B, Λ) be a compatible pair. Then $B \cdot \Lambda$ has full rank.*

Let (B, Λ) be a compatible pair and let $k \in [1, m]$. We define for $\varepsilon \in \{+1, -1\}$ a $n \times n$ matrix $E_{k, \varepsilon}$ via

- $(E_{k, \varepsilon})_{ij} = \delta_{ij}$ if $j \neq k$,
- $(E_{k, \varepsilon})_{ij} = -1$ if $i = j = k$,
- $(E_{k, \varepsilon})_{ij} = \max(0, -\varepsilon b_{ki})$ if $i \neq j = k$.

Similarly, we define a $m \times m$ matrix $F_{k, \varepsilon}$ via

- $(F_{k, \varepsilon})_{ij} = \delta_{ij}$ if $i \neq k$,
- $(F_{k, \varepsilon})_{ij} = -1$ if $i = j = k$,
- $(F_{k, \varepsilon})_{ij} = \max(0, \varepsilon b_{jk})$ if $i = k \neq j$.

We define a new pair (B_k, Λ_k) as

$$(2.8) \quad B_k = F_{k, \varepsilon} B E_{k, \varepsilon} , \quad \Lambda_k = E_{k, \varepsilon}^T \Lambda E_{k, \varepsilon} ,$$

where X^T denotes the transpose of X . We have the following fact.

Proposition 2.13. [2, Prop. 3.4] *The pair (B_k, Λ_k) is compatible. Moreover, Λ_k is independent of the choice of the sign ε .*

Proposition 2.10 has the following obvious corollary.

Corollary 2.14. *Let \mathbb{A} be a cluster algebra given by an initial seed (\mathbf{x}, B) where B is a $m \times n$ -matrix. If (B, Λ) is a compatible pair, then Λ defines a compatible Poisson bracket on \mathfrak{F} and $\mathcal{U}(\mathbb{A})$.*

Example 2.15. *If $m = n$ (i.e. there are no coefficients/frozen variables) and B has full rank, then $(B, \mu B^{-1})$ is a compatible pair for all $\mu \in \mathbb{Z}_{>0}$ such that μB^{-1} is an integer matrix.*

Remark 2.16. *Another important example is the following. Recall that double Bruhat cells in complex semisimple connected and simply connected algebraic groups have a natural structure of an upper cluster algebra (see [1]). Berenstein and Zelevinsky showed that the standard Poisson structure is given by compatible pairs relative to this upper cluster algebra structure (see [2, Section 8]).*

2.8. Our running example.

2.8.1. The standard Poisson structure. In the case of $\mathbb{C}[G(2, 5)]$ we have the so called standard Poisson structure which is compatible with the cluster algebra structure. It is most easily defined as the restriction of the standard Poisson bracket on $\mathbb{C}[Mat_{2,5}]$ to the subalgebra $\mathbb{C}[G(2, 5)]$: The standard Poisson bracket on $\mathbb{C}[Mat_{2,n}]$ is given by

$$\{x_{ij}, x_{k\ell}\} = (\text{sgn}(i - k) + \text{sgn}(j - \ell)) x_{i\ell} x_{kj} ,$$

where $i, j \in \{1, 2\}$ and $k, \ell \in [1, n]$ and sgn denotes the sign function.

We observe that the Poisson bracket in the cluster

$$(\Delta_{13}, \Delta_{14}, \Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{45}, \Delta_{15})$$

is given by the matrix

$$(2.9) \quad \Lambda = \begin{pmatrix} \mathbf{0} & -\mathbf{1} & 1 & -1 & -1 & -2 & -1 \\ \mathbf{1} & \mathbf{0} & 1 & 0 & -1 & -2 & -1 \\ -1 & -1 & 0 & -1 & -2 & -2 & -1 \\ 1 & 0 & 1 & 0 & -1 & -2 & 0 \\ 1 & 1 & 2 & 1 & 0 & -1 & 0 \\ 2 & 1 & 2 & 2 & 1 & 0 & 1 \\ 1 & 1 & 1 & 0 & 0 & -1 & 0 \end{pmatrix} .$$

It can be verified by direct computation that (B, Λ) is a compatible pair.

2.8.2. The toric actions. The torus actions on the cluster algebra are given by the usual torus actions on the Grassmannian, i.e. the action of $(\mathbb{C}^*)^2 \times (\mathbb{C}^*)^5$ on $M \in Mat_{2,5}$ via left-, resp. right-multiplication by diagonal matrices:

$$\begin{pmatrix} \ell_1 & 0 \\ 0 & \ell_2 \end{pmatrix} \cdot M \cdot \begin{pmatrix} r_1 & 0 & 0 & 0 & 0 \\ 0 & r_2 & 0 & 0 & 0 \\ 0 & 0 & r_3 & 0 & 0 \\ 0 & 0 & 0 & r_4 & 0 \\ 0 & 0 & 0 & 0 & r_5 \end{pmatrix} .$$

The weights of these actions on the initial cluster are:

$$wt(\ell_1) = wt(\ell_2) = (1, 1, 1, 1, 1, 1, 1), \quad wt(r_1) = (1, 1, 1, 0, 0, 0, 1), \quad wt(r_2) = (0, 0, 1, 1, 0, 0, 0),$$

$$wt(r_3) = (1, 0, 0, 1, 1, 0, 0), \quad wt(r_4) = (0, 1, 0, 0, 1, 1, 0), \quad wt(r_5) = (0, 0, 0, 0, 0, 1, 1) .$$

It is easy to verify that the weights span the kernel of B .

3. TORIC POISSON PRIME IDEALS

3.1. The Main Theorem. In this section we will prove the main result of the paper. The main theorem requires our seeds to be generic in some sense. However, the parts of the proof each hold in some more generality, and we will need the more general versions later on. Therefore, we will introduce our restrictions one by one throughout the section. First, we recall the definitions of Poisson, Poisson prime and toric Poisson prime ideals.

Definition 3.1. Let $(A, \{\cdot, \cdot\})$ be a Poisson algebra over \mathbb{C} and T an algebraic torus acting on \mathbb{A} by automorphisms.

- (a) A Poisson ideal $\mathcal{I} \subset A$ is an ideal which is closed under the Poisson bracket; i.e., $\{h, a\} \in \mathcal{I}$ if $h \in \mathcal{I}$ and $a \in \mathbb{A}$.
- (b) A Poisson prime ideal is a Poisson ideal which is prime.
- (c) A toric Poisson prime ideal (TPP for short) is a Poisson ideal which is prime and torus invariant.

Now suppose that $\mathbf{x} = (x_1, \dots, x_n)$ and that B is a $m \times n$ -integer matrix with skew-symmetrizable principal part (i.e. the corresponding cluster algebra has rank n and $n - m$ coefficients). Let Λ be a skew-symmetric integer matrix such that (\mathbf{x}, B, Λ) defines a Poisson cluster algebra. We have the following main result (the precise statement follows in Theorem 3.29).

Main Theorem 3.2. Let \mathfrak{A} be a Noetherian Poisson cluster algebra over the complex numbers, corresponding to the triple (\mathbf{x}, B, Λ) introduced above. Let X be an affine variety such that $\mathfrak{A} = \mathbb{C}[X]$ is its coordinate ring. If the seed is generic (for precise definitions see below), then \mathbb{A} contains only finitely many Poisson prime ideals which are invariant under the global toric actions.

Remark 3.3. The proof also works if \mathbb{A} is a Noetherian upper cluster algebra or an upper bound.

Remark 3.4. Recall that if (B, Λ) is a compatible pair, then $B \cdot \Lambda$ has full rank (see Lemma 2.12).

We observe the following fact.

Proposition 3.5. Let \mathbb{A} be a Noetherian Poisson cluster algebra given by (\mathbf{x}, B, Λ) , and let $(\mathbf{x}, B, \Lambda')$ define another compatible Poisson structure. If \mathcal{I} is a TPP for (\mathbf{x}, B, Λ) , then it is also a TPP for $(\mathbf{x}, B, \Lambda')$.

Proof. By Proposition 2.10 it suffices to show that if $t \in T$ is an element of the torus of global toric actions and $f \in \mathcal{I}$, then $t.f \in \mathcal{I}$, as well. But this is the definition of torus invariant. \square

Additionally, notice the following fact.

Lemma 3.6. Let \mathfrak{A} be a Noetherian cluster algebra over the complex numbers, and let $\{\cdot, \cdot\}$ be a compatible Poisson structure. Then, the global toric actions are compatible with the Poisson structure, i.e.

$$\{t.x, t.y\} = t.\{x, y\}, \quad \text{for all } x, y \in \mathbb{A}.$$

Proof. Express x and y as Laurent polynomials in a cluster. The assertion is now proved by straightforward computation, using the fact that the Poisson bracket is log-canonical in the cluster variables. \square

Let us return to Theorem 3.2 which we will prove in four steps which contain several independent and important results.

3.2. Step 1: The set TP_Y . Let \mathbf{x} be a cluster. We will, for convenience, also use the notation \mathbf{x} to denote the set $\mathbf{x} = \{x_1, \dots, x_n\}$. Denote, as before, by y_1, \dots, y_m the cluster variables obtained by mutation in the directions $1, \dots, m$, respectively. Let

$$Y = \{x_1, \dots, x_n, y_1, \dots, y_m\} \cup \{1\}.$$

Denote by I_S the Poisson ideal generated by a subset $S \subset Y$. Notice that if $1 \in S$ then, of course, $I_S = \mathbb{A}$. Denote by $J_S \subset \mathbb{C}[x_1, \dots, x_n]$ the ideal generated by $S \subset \mathbf{x}$ in $\mathbb{C}[x_1, \dots, x_n]$. The ideal J_S is torus invariant Poisson and prime in the polynomial ring $\mathbb{C}[x_1, \dots, x_n]$. Denote by M_S the monoid consisting of all monomials in $\{x_{j_1}, \dots, x_{j_i}\} = \mathbf{x} - S$; i.e., a typical element of M_S is a monomial x^β where $0 \neq \beta \in \mathbb{Z}_{\geq 0}^n$ and $\beta_i = 0$ if $x_i \in S$.

We define a subset TP_Y of the power set of Y as follows.

Definition 3.7. *The set TP_Y is the set whose elements are the subsets of $S \subset Y$ such that*

- (1) $I_S \cap Y = S$,
- (2) $I_S \cap \mathbb{C}[x_1, \dots, x_n] = J_S$.

Definition 3.7 implies the following lemma.

Lemma 3.8. *Let $S \in TP_Y$. Then,*

- (1) $I_S \cap M_S = \emptyset$, and
- (2) $x_i \in S$ or $y_i \in S$ is equivalent to $P_i \in J_S$.

Remark 3.9. *When determining the elements of TP_Y , it is clearly most difficult to verify that $I_S \cap Y = S$. However, we conjecture (see Conjecture 5.9) that we can solve this problem combinatorially, using the concept of defining clusters introduced in Section 3.4.*

3.3. Step 2: TP_Y characterizes TPPs. We begin with the following key result. Recall the notation $\ker(B) = T$.

Proposition 3.10. *Let \mathbf{x} be a cluster, $\text{rank}(T + \text{Im}(\Lambda)) = n$ and \mathcal{I} be a non-zero TPP. Then the ideal \mathcal{I} contains a cluster variable $x_i \in \mathbf{x}$.*

Remark 3.11. *The condition $\text{rank}(T + \text{Im}(\Lambda)) = n$ is satisfied if (B, Λ) is a compatible pair. Indeed, by definition $\text{rank}(B \cdot \Lambda) = m$, resp. $\text{rank}(B(\text{Im}(\Lambda))) = m$, and, hence, $\text{rank}(T + \text{Im}(\Lambda)) = n$.*

Proof. Notice first that $\mathcal{I}_{\mathbf{x}} \neq 0$. Indeed, let $0 \neq f \in \mathcal{I}$. We can express f as a Laurent polynomial in the variables x_1, \dots, x_n ; i.e., $f = x_1^{-c_1} \dots x_n^{-c_n} g$ where $c_1, \dots, c_n \in \mathbb{Z}_{\geq 0}$ and $0 \neq g \in \mathbb{C}[x_1, \dots, x_n]$. Clearly, $g = x_1^{c_1} \dots x_n^{c_n} f \in \mathcal{I}_{\mathbf{x}}$. Observe, additionally, that $\mathcal{I}_{\mathbf{x}}$ is prime and torus invariant.

We complete the proof by contradiction. Let $f = \sum_{\mathbf{w} \in \mathbb{Z}^n} c_{\mathbf{w}} x^{\mathbf{w}} \in \mathcal{I}_{\mathbf{x}}$ and suppose that f cannot be factored into $f = gh$ with $g \in \mathcal{I}_{\mathbf{x}}$ or $h \in \mathcal{I}_{\mathbf{x}}$. We have to show that $f = x_i$ for some i . Since the ideal is prime, it suffices to show that f is a monomial. We assume that f has the smallest number of nonzero summands such that no monomial term $c_{\mathbf{w}} x^{\mathbf{w}}$ with $c_{\mathbf{w}} \neq 0$ is contained in \mathcal{I} . It must therefore have at least two monomial terms.

We need the following fact.

Lemma 3.12. *Using the notation introduced above, a monomial $x^{\mathbf{w}}$ with $\mathbf{w} \in \mathbb{Z}^n$ is torus invariant if and only if $\mathbf{w} \in T^\top$, where $^\top$ denotes the orthogonal complement with respect to the standard bilinear form on \mathbb{Z}^n .*

Proof. Recall from Section 2.5 that if $\mathbf{b} \in T$ defines a global toric action $\psi_{\mathbf{x}, \alpha}$, then $x^{\mathbf{w}}$ is invariant under $\psi_{\mathbf{x}, \alpha}$ if and only if $\sum_{i=1}^n w_i b_i = 0$. The assertion follows. \square

The function f , considered above, must be torus invariant, hence for each pair $\mathbf{w}, \mathbf{w}' \in \mathbb{Z}^n$ with $c_{\mathbf{w}}, c_{\mathbf{w}'} \neq 0$ we obtain that $\mathbf{w} - \mathbf{w}' = \mathbf{v} \in T^\top$. Denote by $\text{rad}(\Lambda)$ the radical of the skew-symmetric bilinear form, i.e. the set of $\mathbf{u} \in \mathbb{Z}^n$ such that $\mathbf{u}^T \cdot \Lambda \cdot \mathbf{u}' = 0$ for all $\mathbf{u}' \in \mathbb{Z}^n$. We have the following fact.

Lemma 3.13. *The intersection $\text{rad}(\Lambda) \cap T^\top = \{0\}$.*

Proof.

We know that $\text{Im}(\Lambda) \otimes \mathbb{Q} + \ker(B) \otimes \mathbb{Q} = \mathbb{Q}^n$. Hence $\text{Im}(\Lambda)^\top \otimes \mathbb{Q} \cap T \otimes \mathbb{Q}^\top = \{0\}$. Notice, that by definition $\text{Im}(\Lambda)^\top \otimes \mathbb{Q} = \text{rad}(\Lambda) \otimes \mathbb{Q}$. The assertion is proved. \square

Assume as above that $c_{\mathbf{w}}, c_{\mathbf{w}'} \neq 0$ and $\mathbf{w} - \mathbf{w}' = \mathbf{v} \in T^\top$. The previous lemma yields that $\mathbf{v} \notin \text{rad}(\Lambda)$. This implies that there exists $i \in [1, n]$ such that $\{x_i, x^{\mathbf{v}}\} \neq 0$. Therefore, $\{x_i, x^{\mathbf{w}}\} = cx_i x^{\mathbf{w}} \neq dx_i x^{\mathbf{w}'} \{x_i, x^{\mathbf{w}'}\}$ for some $c, d \in \mathbb{C}$.

Clearly, $cx_i f - \{x_i, f\} \in \mathcal{I}$ and

$$cx_i f - \{x_i, f\} = (c - c)c_{\mathbf{w}} x^{\mathbf{w}} + (c - d)c_{\mathbf{w}'} x^{\mathbf{w}'} + \dots$$

Hence, $cx_i f - \{x_i, f\} \neq 0$ and it has fewer monomial summands than f which contradicts our assumption. Therefore, \mathcal{I} contains a monomial, and because it is prime it must contain some $x_i \in \mathbf{x}$. The proposition is proved. \square

Let \mathbf{x} be a cluster in \mathbb{A} and $\mathbf{i} = \{x_{i_1}, \dots, x_{i_k}\}$ be a k -element subset of \mathbf{x} . Moreover, denote by $\mathbf{j} = \{x_{j_1}, \dots, x_{j_\ell}\} = \mathbf{x} - \mathbf{i}$ with $\ell = n - k$ and by $\Lambda_{\mathbf{i}}$ the submatrix of Λ obtained by removing the rows and columns labeled by $\{i_1, \dots, i_k\}$. Recall the notation $T_{\mathbf{i}}$ as introduced in Section 2.6. Observe that $\text{Im}(\Lambda_{\mathbf{i}})$ and $T_{\mathbf{i}}$ can be naturally viewed as sublattices of the lattice $\mathbb{Z}^{\mathbf{j}} \subset \mathbb{Z}^n$ generated by the standard basis vectors $e_{j_1}, \dots, e_{j_\ell}$. We need our first major condition.

Condition 3.14. *Let B and Λ be as above. The cluster \mathbf{x} is super-toric, if $\text{rank}(T_{\mathbf{i}} + \text{Im}(\Lambda_{\mathbf{i}})) = n - k = \ell$ for all subsets $\mathbf{i} \subset [1, n]$.*

Remark 3.15. *Notice that if B and Λ are generic, and $m < n$, then the corresponding cluster will be super-toric. Indeed, for generic matrices we have*

$$\text{rank}(\text{Im}(\Lambda_{\mathbf{i}})) = \min(\text{rank}(\Lambda_{\mathbf{i}}), 2 \lfloor \frac{n-k}{2} \rfloor)$$

while $\text{rank}(T_{\mathbf{i}}) = \min(\text{rank}(T), n - k)$.

Theorem 3.16. *Let \mathbb{A} be a Noetherian Poisson cluster algebra and \mathbf{x} a super-toric cluster, and let \mathcal{I} be a TPP. Then the set $\mathcal{I} \cap Y = S$ is an element of TPY .*

Proof.

The first condition of Definition 3.7 is obvious. The second, however, is a bit more interesting. It follows from a stronger version of Proposition 3.10.

Proposition 3.17. *Let \mathbf{x} be a super-toric cluster, and \mathcal{I} be a non-zero TPP. Then the ideal $\mathcal{I}_{\mathbf{x}} = \mathcal{I} \cap \mathbb{C}[x_1, \dots, x_n]$ is generated by a non-empty subset of \mathbf{x} .*

Proof. Suppose that $\mathcal{I} \cap \mathbf{x} = \mathbf{i} = \{x_{i_1}, \dots, x_{i_k}\}$. Suppose that $f \in \mathbb{C}[x_{j_1}, \dots, x_{j_\ell}] \in \mathcal{I}$. The cluster, however, is super-toric, hence we can adapt the argument from the proof of Proposition 3.10 to $\mathbb{C}[x_{j_1}, \dots, x_{j_\ell}]$, $T_{\mathbf{i}}$ and $\Lambda_{\mathbf{i}}$ and show that there exists some $x_j \notin \mathbf{i}$ such that $x_j \in \mathcal{I}$. We obtain the desired contradiction and the proposition is proved. \square

Theorem 3.16 is proved. \square

Proposition 3.17 has the following immediate consequence. Denote by $X_{\mathcal{I}}$ the zero locus of an ideal \mathcal{I} in X .

Proposition 3.18. *Let $\mathcal{I} \subset \mathbb{A}$ be a TPP, \mathbf{x} a super-toric cluster and let $|\mathcal{I} \cap \mathbf{x}| = k$. Then $\dim(X_{\mathcal{I}}) \geq n - k = \ell$.*

Proof. Recall that we denote by $\{x_{j_1}, \dots, x_{j_\ell}\}$ the set $\mathbf{x} - S$. Observe that $\{x_{j_1}, \dots, x_{j_\ell}\}$ is algebraically independent over \mathbb{A}/\mathcal{I} by Proposition 3.10. Moreover, no Laurent polynomial $f = x^{-\mathbf{w}}g$ with $\mathbf{w} \in \mathbb{Z}_{\geq 0}^n$ and $g \in \mathbb{C}[x_{j_1}, \dots, x_{j_\ell}]$ is contained in the ideal. Hence, the field of fractions $\mathbb{C}(x_{j_1}, \dots, x_{j_\ell})$ embeds into the field of fractions of \mathbb{A}/\mathcal{I} and the assertion follows. \square

3.4. Step 3: Existence of TPPs for $S \in TP_Y$. In order to study a TPP \mathcal{I} we need to capture as much information as possible in the set $\mathbf{x} \cap \mathcal{I}$. For this reason we prefer to work with clusters such that the cardinality of $\mathbf{x} \cap \mathcal{I}$ equals the dimension of $X_{\mathcal{I}}$. We therefore introduce the notion of *defining clusters*. Their properties will be further investigated in the following Section 3.5.

Definition 3.19. (a) Let \mathbb{A} be a Poisson cluster algebra and $\mathcal{I} \subset \mathbb{A}$ a TPP. A cluster \mathbf{x} is called a *defining cluster* for \mathcal{I} if $x_i \in \mathcal{I}$ implies that $y_i \in \mathcal{I}$, as well.
(b) A set $S \in TP_Y$ is called *defining*, if $x_i \in S$ implies that $y_i \in S$, as well.

The main result of this section is the following theorem.

Theorem 3.20. *Let \mathbb{A} be as above, and let $S \in TP_Y$ be defining. Then there exists a TPP $\mathcal{I} \subset \mathbb{A}$ such that $\mathcal{I} \cap Y = S$.*

Proof.

We first introduce the notion of a Poisson multiplicative set. Let \mathbb{A} be a Poisson algebra and $S \subset \mathbb{A}$ a multiplicative set (i.e., $s \cdot t \in S$ for all $s, t \in S$). We call S a *Poisson multiplicative set* if $\{s, t\} \in S \cup \{0\}$ for all $s, t \in S$. We have the following fact.

Lemma 3.21. *Let \mathbb{A} be a Poisson algebra and let $S \subset \mathbb{A}$ be a Poisson multiplicative set. (a) The algebra $\mathbb{A}[S^{-1}]$ is a Poisson algebra with Poisson bracket*

$$\{s^{-1}f, t^{-1}g\} = s^{-2}t^{-2}\{s, t\}fg - s^{-2}t^{-1}\{s, g\}f - s^{-1}t^{-2}\{f, t\}g + s^{-1}t^{-1}\{f, g\},$$

for all $s, t \in S$ and $f, g \in \mathbb{A}$.

(b) *The natural embedding of algebras $\mathbb{A} \hookrightarrow \mathbb{A}[S^{-1}]$ is a homomorphism of Poisson algebras.*

(c) *Extension and Contraction define maps from the set of Poisson ideals \mathcal{I} in \mathbb{A} such that $\mathcal{I} \cap S = \emptyset$ to the set of Poisson ideals in $\mathbb{A}[S^{-1}]$. Moreover, if restricted to prime ideals, this map becomes a bijection.*

Proof. Part (a) can be proved by direct computation using the identity $\{s^{-1}, f\} = -s^{-2}\{s, f\}$ for all $s \in S$ and $f \in \mathbb{A}$. It follows from

$$0 = \{1, f\} = \{s^{-1}s, f\} = s^{-1}\{s, f\} + s\{s^{-1}, f\}.$$

For part (b) it suffices to observe that $\mathbb{A} \subset \mathbb{A}[S^{-1}]$ is a Poisson subalgebra. Part(c) is also immediate from standard localization theory and the fact that if $B \subset A$ is a Poisson subalgebra of a Poisson algebra A and $J \subset A$ a Poisson ideal, then the intersection $J \cap B$ is a Poisson ideal in B . The lemma is proved. \square

Recall that we denote by $\{x_{j_1}, \dots, x_{j_\ell}\}$ the set $\mathbf{x} - S$. Observe that the set $\{\gamma x_{j_1}^{\gamma_1} \dots x_{j_\ell}^{\gamma_\ell} : \gamma \in \mathbb{C}; \alpha_1, \dots, \alpha_\ell \in \mathbb{Z}_{\geq 0}\}$ is a Poisson multiplicative set. Consider the ideal \hat{I}_S generated by S in $\mathbb{A}[x_{j_1}^{-1}, \dots, x_{j_\ell}^{-1}]$. It is proper, hence contained in a minimal prime ideal (recall that an ideal P in a ring R is called a minimal prime over an ideal I if P/I is a minimal prime in R/I). It is a Poisson prime ideal by the following fact.

Lemma 3.22. [19, Lemma 6.2] *Let \mathbb{A} be a Poisson algebra over \mathbb{C} and let $\mathcal{I} \subset \mathbb{A}$ be a Poisson ideal. Then all minimal prime ideals over \mathcal{I} are Poisson ideals.*

The ideal \hat{I}_S is torus invariant by Lemma A.3. Its intersection with \mathbb{A} yields a TPP \mathcal{I} . By construction $\mathcal{I} \cap \mathbf{x} = S \cap \mathbf{x}$. We now obtain from Proposition 3.10 that if an exchange polynomial $P_i \in \mathcal{I}$ (and, therefore, $P_i \in J_S$) then $y_i \in S$, because S is defining. We conclude that $\mathcal{I} \cap Y = S$. Theorem 3.20 is proved. \square

3.5. Step 4: Finiteness of the Stratification. In order to prove the finite-ness of the stratification we need to make another assumption, in some sense saying that the cluster variables are generic in a geometric way. This assumption appears to be the hardest to verify for a given cluster algebra.

Condition 3.23. *Let \mathbb{A} and \mathbf{x} be as above.*

(a) *We say that \mathbf{x} is geometrically generic if for all $S \in TP_Y$ for which \mathbf{x} is defining and all minimal Poisson primes P over S we have $\dim(\mathbb{A}/P) = n - |\mathbf{i}|$ where $\mathbf{i} = S \cap \mathbf{x}$.*

(b) *Let $r \in \mathbb{Z}_{\geq 0}$. We say that \mathbf{x} is geometrically r -generic if each cluster that can be reached from r with at most r mutations is geometrically generic.*

Remark 3.24. *A cluster \mathbf{x} is geometrically generic if for each minimal Poisson prime P over some set $S \in TP_Y$ for which \mathbf{x} is defining, there exist prime ideals (not necessarily Poisson) $0 = \mathcal{I}_0 \subsetneq \mathcal{I}_1 \subsetneq \dots \subsetneq \mathcal{I}_k = P$ such that $\mathcal{I}_j \cap \mathbf{x} = \{x_{i_1}, \dots, x_{i_j}\}$ for all $1 \leq j \leq k$.*

Recall that $m \leq n$ is the number of cluster variables in each cluster.

Proposition 3.25. *Let \mathbb{A} and \mathbf{x} be as above, and let $S \in TP_Y$. If \mathbf{x} is geometrically m -generic, then there exist only finitely many TPPs $\mathcal{I}_1, \dots, \mathcal{I}_\ell$ such that $\mathcal{I}_j \cap Y = S$.*

Proof.

We first need some properties of geometrically generic clusters.

Lemma 3.26. *Suppose that \mathbf{x} is geometrically generic and a defining cluster for a TPP $\mathcal{I} \subset \mathbb{A}$ with $\mathcal{I} \cap Y = S$. Then $|\mathbf{x} - S| = \dim(X_{\mathcal{I}})$.*

Proof. Proposition 3.18 yields that $|\mathbf{x} - S| \leq \dim(\mathbb{A}/\mathcal{I})$. Equality follows from the assumption that the cluster is geometrically generic. The lemma is proved. \square

Starting from any cluster \mathbf{x} we can construct a defining cluster for a TPP $\mathcal{I} \subset \mathbb{A}$ using the following algorithm.

Algorithm 3.27. (a) Start with \mathbf{x} and choose, if possible, one i such that $x_i \in S$ and $y_i \notin S$. If there exists no such i , then the algorithm terminates.

(b) Consider the cluster $\mathbf{x}_i = (x_1, \dots, \hat{x}_i, y_i, \dots, x_n)$ and the set S_i , defined for \mathbf{x}_i , just as S is defined for \mathbf{x} .

(c) Repeat Step (a) with $\mathbf{x} = \mathbf{x}_i$.

The algorithm terminates after at most m iterations, and we obtain a cluster \mathbf{x}' such that $\mathcal{I} \cap Y'$ is defining.

From now on, we may therefore assume that \mathbf{x} is defining for a TPP \mathcal{I} with $\mathcal{I} \cap Y = S$.

Lemma 3.28. *The ideal \mathcal{I} is a minimal prime over I_S .*

Proof. Suppose that \mathcal{I} is not minimal over I_S and let \mathcal{I}' be a minimal prime over I_S such that $\mathcal{I} \supset \mathcal{I}'$. Then $\dim(X_{\mathcal{I}}) < \dim(X_{\mathcal{I}'}) \leq |S \cap \{\mathbf{x}\}|$. This, however, contradicts Proposition 3.18. The lemma is proved. \square

Now, let \mathbf{x} be any cluster and Y as above. There are only finitely many clusters $\mathbf{x}_1, \dots, \mathbf{x}_p$ that one can reach from \mathbf{x} with at most m mutations, each of which is geometrically generic by our assumptions. Denote by Y_1, \dots, Y_p the corresponding sets " Y ". Each TPP is a minimal prime over some I_T where $T \subset Y_h$ for some $1 \leq h \leq p$. The union of the power sets of Y_1, \dots, Y_p is finite. We assumed \mathbb{A} to be Noetherian, hence there exist only finitely many minimal prime ideals over any given ideal I_T . Proposition 3.25 is proved. \square

Hence, we have proved Theorem 3.2, in the following precise form.

Theorem 3.29. *Let \mathfrak{A} be a Noetherian Poisson cluster algebra over the complex numbers, corresponding to the triple (\mathbf{x}, B, Λ) introduced above. If the seed is super-toric and geometrically m -generic, then \mathbb{A} contains only finitely many Poisson prime ideals which are invariant under the global toric actions.*

3.6. Torus invariant Poisson prime ideals in $\mathbb{C}[G(2, 5)]$. The set Y we have to consider is

$$Y = \{\Delta_{13}, \Delta_{24}, \Delta_{14}, \Delta_{35}, \Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{45}, \Delta_{15}, \Delta_{25}\}.$$

The TPPs that have co-dimension one are generated by the coefficients $\Delta_{12}, \Delta_{23}, \Delta_{34}, \Delta_{45}, \Delta_{15}$. Notice that the cluster variables which are not coefficients cannot generate Poisson ideals, as e.g.

$$\{\Delta_{13}, \Delta_{24}\} = 2\Delta_{14}\Delta_{23}.$$

Now, consider $S = \{\Delta_{12}, \Delta_{23}\} \subset Y$. S does not define a toric Poisson prime ideal, since an ideal that contains S but neither Δ_{13} nor Δ_{24} cannot be prime because of the Plücker relation (Equation 2.4)

$$\Delta_{13}\Delta_{24} = \Delta_{14}\Delta_{23} + \Delta_{12}\Delta_{34} \in \mathcal{I}.$$

However, one easily verifies that $S_1 = \{\Delta_{12}, \Delta_{23}, \Delta_{13}\}$, $S_2 = \{\Delta_{12}, \Delta_{23}, \Delta_{24}\}$ and $S_3 = \{\Delta_{12}, \Delta_{23}, \Delta_{13}, \Delta_{24}\}$ define toric Poisson prime ideals. Observe that the first two have co-dimension two, while the third one has co-dimension three.

3.7. Varieties with singularities. Recall that the cluster manifold defined by Gekhtman, Shapiro and Vainsthein in [15, Section 2.1] is smooth. If the variety X defining a cluster algebra $\mathbb{A} = \mathbb{C}[X]$ is singular, then the cluster manifold will be a smooth submanifold. Notice, however, that the singular points are the zero locus of a Poisson ideal (see [45]). Thus, we are able to recover the whole singular variety, as the following example shows which is adapted from an example in [15].

Example 3.30. Consider a cluster algebra \mathbb{A} over \mathbb{C} defined by two clusters (x_1, x_2, x_3) and (x'_1, x_2, x_3) with exchange relation $x_1 x'_1 = x_2^2 + x_3^2$ and compatible Poisson structure

$$\{x_1, x_2\} = x_1 x_2, \quad \{x_1, x_3\} = -x_1 x_3, \quad \{x_2, x_3\} = 0.$$

We have $\mathbb{A} \cong \mathbb{C}[a, b, c, d]/(ab = c^2 + d^2)$, which defines a hypersurface X in \mathbb{C}^4 with a singularity at $a = b = c = d = 0$. Now let us determine the quotient cluster algebras. It is easy to see that there are the following toric Poisson ideals (we only list the generators) $\langle x_2 \rangle$, $\langle x_3 \rangle$, $\langle x_2, x_3 \rangle$, $\langle x_1, x_2, x_3 \rangle$, $\langle x'_1, x_2, x_3 \rangle$ and $\langle x_1, x'_1, x_2, x_3 \rangle$. Gekhtman, Shapiro and Vainsthein show in [15] that $X \setminus \{(0, 0, 0, 0)\}$ is the cluster manifold. In our picture, we observe that the singularity $(0, 0, 0, 0)$ defines a toric Poisson ideal in \mathfrak{A} .

4. ACYCLIC CLUSTER ALGEBRAS

We will now apply Theorem 3.2 to the case of certain acyclic cluster algebras, namely those which have a seed (\mathbf{x}, B) where B is a skew-symmetric $n \times n$ -matrix with $b_{ij} > 0$ if $i < j$. Berenstein, Fomin and Zelevinsky proved in [1] that such a cluster algebra \mathbb{A} is equal to both its lower and upper bounds. Thus, it is Noetherian and, if B has full rank, a Poisson algebra with the Poisson brackets given by compatible pairs (B, Λ) with $\Lambda = \mu B^{-1}$ for certain $\mu \in \mathbb{Z}$. In order for B to have full rank we have to assume that $n = 2k$ is even. Finally, notice that there are no global toric actions as $\ker(B)$ is trivial. Hence all Poisson prime ideals are TPPs. Let $P_i = m_i^+ + m_i^-$ where m_i^+ and m_i^- denote the monomial terms in the exchange polynomial. Then $\{y_i, x_i\} = \mu_1 m_i^+ + \mu_2 m_i^-$ for some $\mu_1, \mu_2 \in \mathbb{Z}$. We, additionally, want to require that $\mu_1 \neq \mu_2$. To assure this, we assume that

$$(4.1) \quad \sum_{j=1}^n (b^{-1})_{ij} (\max(b_{ij}, 0) + \min(b_{ij}, 0)) \neq 0$$

for all $i \in [1, n]$. We have the following result.

Theorem 4.1. Let \mathbb{A} be an acyclic cluster algebra over \mathbb{C} with trivial coefficients of even rank $n = 2k$, given by a seed (x_1, \dots, x_n, B) where B is a skew-symmetric $n \times n$ -integer matrix satisfying $b_{ij} > 0$ if $i < j$ and suppose that B and B^{-1} satisfy Equation 4.1 for each $i \in [1, n]$. Then, the Poisson cluster algebra defined by a compatible pair (B, Λ) where $\Lambda = \mu B^{-1}$ with $0 \neq \mu \in \mathbb{Z}$ contains no non-trivial Poisson prime ideals.

Proof. Suppose that there exists a non-trivial TPP or Poisson prime ideal \mathcal{I} . Then, $\mathcal{I} \cap \mathbf{x}$ is nonempty by Proposition 3.10, hence $\mathcal{I} \cap \mathbf{x} = \{x_{i_1}, \dots, x_{i_j}\}$ for some $1 \leq i_1 \leq i_2 \leq \dots \leq i_j \leq 2k$. Note that \mathbf{x} does not need to be defining for the ideal

\mathcal{I} . Observe that if $b_{i_1, h} < 0$, then $x_h \notin \mathcal{I}$ for all $1 \leq h \leq n$. Additionally, observe that $P_{i_1} = m_{i_1}^+ + m_{i_1}^-$ has to be contained in \mathcal{I} , as well as

$$\{y_{i_1}, x_{i_1}\} = \mu_1 m_{i_1}^+ + \mu_2 m_{i_1}^-.$$

By our assumption, we have $\mu_1 \neq \mu_2$, and therefore $m_{i_1}^- \in \mathcal{I}$. Hence, $x_h \in \mathcal{I}$ for some $h \in [1, i_1 - 1]$ or $1 \in \mathcal{I}$. We obtain the desired contradiction and the theorem is proved. \square

The theorem has the following corollary which was also independently proved by Muller very recently [39], though in more generality.

Corollary 4.2. *Let \mathbb{A} be as in Theorem 4.1. Then, the variety X defined by $\mathbb{A} = \mathbb{C}[X]$ is smooth.*

Proof. The singular subset is contained in a Poisson ideal of co-dimension greater or equal to one (see Section 3.7). The Poisson ideal must be contained in a proper Poisson prime ideal by Lemma 3.22. The assertion follows. \square

Remark 4.3. *The assumption that the cluster algebra has even rank is very important. Indeed, Muller has recently shown that the variety corresponding to the cluster algebra of type A_3 has a singularity ([38, Section 6.2]).*

Remark 4.4. *We believe that our results also extend to the locally acyclic cluster algebras introduced in [39]. However, it should be possible to show that the variety has additionally the structure of a (holomorphic) symplectic manifold.*

5. EXPLICIT DESCRIPTION OF IDEALS AND COS

In the following section we make some additional assumptions about the cluster algebras in questions and their TPPs. In particular, we assume the following *COS* Condition. We use the terminology of a defining cluster, introduced in Section 3.5. We will also refer to a Noetherian Poisson cluster algebra, simply as a cluster algebra.

Condition 5.1. Codimension One Strata (COS) *Let \mathbb{A} be a cluster algebra of rank n , resp. upper cluster algebra or upper bound. We say that \mathbb{A} satisfies COS if for each $1 \leq k \leq \ell \leq n$ and each pair of TPPs $\mathcal{I} \subset \mathcal{I}'$ of co-dimension $n - \ell$, resp. $n - k$, there exists a chain of of TPPs $\mathcal{I} = \mathcal{I}_0 \subsetneq \mathcal{I}_1 \subsetneq \dots \subsetneq \mathcal{I}_{k-\ell} = \mathcal{I}'$.*

Remark 5.2. *Once again, we will ignore the case of upper cluster algebras, resp. upper bounds, however, the arguments are analogous.*

Notice that the condition implies the following.

Lemma 5.3. *Let \mathbb{A} be a cluster algebra, satisfying COS with a geometrically m -generic cluster \mathbf{x} . For each pair of TPPs $\mathcal{I} \subsetneq \mathcal{I}'$ there exists a defining cluster \mathbf{x}' and a sequence $(x'_{i_1}, x'_{i_2}, \dots, x'_{i_k}) \in \mathbf{x}'$, as well as TPPs $\mathcal{I} = \mathcal{I}_0 \subsetneq \mathcal{I}_1 \subsetneq \dots \subsetneq \mathcal{I}_k = \mathcal{I}'$ for which \mathbf{x}' is defining, such that for all $j \in [1, k]$*

$$(5.1) \quad \mathcal{I}_j \cap \mathbf{x}' = (\mathcal{I} \cap \mathbf{x}) \cup \{x'_{i_1}, x'_{i_2}, \dots, x'_{i_j}\}.$$

Proof. The assertion follows from the following observation: Let \mathcal{I} and \mathcal{J} be two TPPs such that $\mathcal{I} \subset \mathcal{J}$ and let \mathbf{x} be an m -generic cluster. We can construct a cluster that is defining for both \mathcal{J} and \mathcal{I} by first constructing a defining cluster

\mathbf{x}'' for \mathcal{I} using Algorithm 3.27 and, afterwards constructing a defining cluster for \mathcal{J} . Indeed, if $x_i'' \in \mathcal{J}$ but $y_i'' \notin \mathcal{J}$ for some $x_i'' \in \mathbf{x}''$, then $x_i'' \notin \mathcal{I}$ and $y_i'' \notin \mathcal{I}$, because \mathbf{x}'' is defining for \mathcal{I} . Hence, all the clusters obtained while constructing a defining cluster for \mathcal{J} from \mathbf{x}'' will also be defining for \mathcal{I} . Equation 5.1 now follows from Proposition 3.26 and the fact that all clusters constructed are geometrically generic as this implies that $\dim(\mathbb{A}/\mathcal{I}) = |\mathbf{x}'' - \mathcal{I}|$, resp. $\dim(\mathbb{A}/\mathcal{J}) = |\mathbf{x}'' - \mathcal{J}|$. The lemma is proved. \square

Now, let \mathbb{A} be a cluster algebra, \mathcal{I} a TPP and \mathbf{x} a defining cluster. Let $\mathcal{I} \cap \mathbf{x} = \{x_{i_1}, x_{i_2}, \dots, x_{i_k}\}$ and let $\{x_{j_1}, \dots, x_{j_\ell}\} = \mathbf{x} - \mathcal{I}$. Let $\{j_1, \dots, j_p\}$ be a p -element subset of $\{1, \dots, n\}$. Notice that the $p \times p$ -submatrix $\Lambda_{j_1, \dots, j_p}$ of the Poisson coefficient matrix Λ spanned by the rows and columns labeled by $\{j_1, \dots, j_p\}$ defines a Poisson bracket on $\mathbb{C}[x_{j_1}^{\pm 1}, \dots, x_{j_p}^{\pm 1}]$. We have the following main result.

Theorem 5.4. (a) Let \mathbb{A} be as above satisfying COS, and let \mathbf{x} be a defining cluster for the TPP \mathcal{I} and let \mathbf{x} be geometrically generic and super-toric. Let $z = \sum_{\alpha \in \mathbb{Z}^n} c_\alpha x^\alpha \in \mathbb{A}$, where $c_\alpha \in \mathbb{C}$. We have $z \in \mathcal{I}$ if and only if $c_\alpha \neq 0$ implies that $\alpha_{i_j} \neq 0$ for some $j \in [1, k]$.

(b) There exists an injective Poisson algebra homomorphism

$$\mathbb{A}/\mathcal{I} \hookrightarrow \mathbb{C}[x_{j_1}^{\pm 1}, \dots, x_{j_\ell}^{\pm 1}]$$

with Poisson bracket given by $\Lambda_{j_1, \dots, j_\ell}$, which sends the image of x_{j_r} in \mathbb{A}/\mathcal{I} to $x_{j_r} \in \mathbb{C}[x_{j_1}^{\pm 1}, \dots, x_{j_\ell}^{\pm 1}]$ for all $1 \leq r \leq \ell$.

Remark 5.5. Notice that it is not at all clear that the set of part(a) should even be an ideal.

Proof. We prove the assertion by induction on k . It is trivially satisfied for $k = 0$. Suppose now that the theorem holds for all TPPs for which the intersection with a defining cluster has cardinality less than $k - 1$. In order to simplify notation, and using the fact that the cluster is geometrically generic, we suppose that $\mathcal{I}_{k-1} \cap \mathbf{x} = \{x_{\ell+1}, x_{\ell+2}, \dots, x_n\}$ and $\mathcal{I}_k \cap \mathbf{x} = \{x_\ell, x_{\ell+1}, \dots, x_n\}$ (this might imply that we have to reorder the cluster variables in such a way that not all the coefficients are the "last" indices). We now make the following claim.

Claim 5.6. There are injective homomorphisms of algebras

$$\mathbb{A}/\mathcal{I}_{k-1} \hookrightarrow \mathbb{C}[x_1^{\pm 1}, \dots, x_\ell^{\pm 1}, x_{\ell+1}] \subset \mathbb{C}[x_1^{\pm 1}, \dots, x_{\ell+1}^{\pm 1}].$$

Proof. The second inclusion is trivial. Let us prove the first one. Suppose not. Then there exists $z \in \mathbb{A}/\mathcal{I}_{k-1}$ which can be expressed as $z = x_{\ell+1}^{-k} \sum_{\alpha \in \mathbb{Z}^{\ell+1}} c_\alpha x^\alpha$, where $k \in \mathbb{Z}_{>0}$, $c_\alpha \in \mathbb{C}$ and where k is minimal with the property that $c_\alpha \neq 0$ implies that $\alpha_{\ell+1} \geq 0$. If we multiply by the smallest common denominator (a monomial $x_{\ell+1}^k x_1^{\beta_1} \dots x_\ell^{\beta_\ell}$ with $\beta_1, \dots, \beta_\ell \in \mathbb{Z}_{\geq 0}$), then we obtain an element $\tilde{z} \in \mathbb{C}[x_1, \dots, x_{\ell+1}] \subset \mathbb{A}/\mathcal{I}_{k-1}$. Clearly $\tilde{z} \in \mathcal{I}_k \subset \mathbb{A}/\mathcal{I}_{k-1}$, where we abuse notation and denote by \mathcal{I}_k the image of the TPP $\mathcal{I}_k \subset \mathbb{A}$ in $\mathbb{A}/\mathcal{I}_{k-1}$. The element \tilde{z} contains at least one monomial summand $c_\gamma x^\gamma$, $\gamma \in \mathbb{Z}_{\geq 0}^n$ where $\gamma_{\ell+1} = 0$. We obtain that its pre-image \tilde{z} under the projection map $\mathbb{A} \rightarrow \mathbb{A}/\mathcal{I}_{k-1}$ (which is the identity map on $\mathbb{C}[x_1, \dots, x_{\ell+1}]$) lies in \mathcal{I}_k , and, hence, as \mathbf{x} is super-toric, there exists, by Proposition 3.17, a cluster variable x_i with $i \leq \ell$ such that $x_i \in \mathcal{I}_k$. That however, contradicts our assumption. The claim is proved. \square

Now, consider the ideal generated by $x_{\ell+1}$ in $\mathbb{C}[x_1^{\pm 1}, \dots, x_{\ell}^{\pm 1}, x_{\ell+1}]$ with Poisson structure defined by $\Lambda_{j_1, \dots, j_{\ell+1}}$. It is easy to see that it is torus invariant, Poisson and prime. Consider its intersection $\tilde{\mathcal{I}}$ with $\mathbb{A}/\mathcal{I}_{k-1}$. It suffices to show that it is the unique minimal toric Poisson prime ideal containing the ideal $\hat{\mathcal{I}}$ generated by $x_{\ell+1}$, but none of the x_1, \dots, x_{ℓ} . It is, clearly, toric, Poisson and prime since it is the intersection of a toric Poisson prime ideal with a torus invariant subring. Now, let $z \in \tilde{\mathcal{I}} - \hat{\mathcal{I}}$. Then, there exists a monomial $m = x_1^{\alpha_1} \dots x_{\ell}^{\alpha_{\ell}}$ with $\alpha_1, \dots, \alpha_{\ell} \in \mathbb{Z}_{\geq 0}$ (e.g. the smallest common denominator) such that $mz \in \hat{\mathcal{I}}$. Hence any TPP \mathcal{J} not containing any of the x_1, \dots, x_{ℓ} must contain z . This implies that $\tilde{\mathcal{I}} \subset \mathcal{J}$, and hence, $\tilde{\mathcal{I}}$ is the unique minimal toric Poisson prime with $\tilde{\mathcal{I}} \cap \mathbf{x} = \{x_{\ell+1}, \dots, x_n\}$. Part (a) is proved and part (b) follows from the fact that if R is a Poisson algebra, $S \subset R$ a Poisson subalgebra and $I \subset R$ a Poisson ideal, then $I \cap S$ is a Poisson ideal in S , and the canonical inclusion $S/(I \cap S) \hookrightarrow R/I$ is a homomorphism of Poisson algebras. The theorem is proved. \square

Theorem 5.4 and Lemma 5.3 imply the following corollary.

Corollary 5.7. *Let \mathbb{A} be a cluster algebra and suppose that all clusters are geometrically generic and super-toric. Let \mathcal{I} , and \mathcal{J} be two TPPs. Then, $\mathcal{I} \subset \mathcal{J}$ if and only if there exists a cluster \mathbf{x} , defining for both \mathcal{I} and \mathcal{J} , such that $(\mathcal{I} \cap \mathbf{x}) \subset (\mathcal{J} \cap \mathbf{x})$.*

5.1. Cluster Algebras satisfying COS and the COS Conjecture. Theorem 5.4 applies in many important cases. It is, for example, well known that the stratification of a complex semisimple connected and simply connected algebraic group G with the standard Poisson structure, into double Bruhat cells provides a stratification that satisfies COS (see e.g. [1] or [2]). The double Bruhat cells $G^{u,v}$ are labeled by double words $u, v \in W$ where W denotes the Weyl group of G . The dimension of $G^{u,v}$ is $\ell(u) + \ell(v) + r$ where $\ell(w)$ denotes the length of an element w of W . The zero loci of the corresponding TPP $J^{u,v}$ are the double Bruhat cells $G^{u',v'}$ where $u' \leq u$ and $v' \leq v$ in the Bruhat order. Indeed, each double Bruhat cell has an upper cluster algebra structure by [1], however, it is not known how to relate cluster algebra structures of different double Bruhat cells. Now, let $u, v \in W$ and let \mathbf{u}, \mathbf{v} be reduced expressions of u and v , respectively. Let \mathbf{w}_0 and \mathbf{w}_0' be reduced expressions of the longest element of the Weyl w_0 such that $\mathbf{w}_0 = \mathbf{u}\mathbf{u}'$ and $\mathbf{w}_0' = \mathbf{v}\mathbf{v}'$ (i.e. the reduced expressions start with \mathbf{u} , resp. \mathbf{v}). It is now easy to see that in the corresponding cluster consisting of generalized minors (see [1]) we obtain the description of $J^{u,v}$ of Theorem 5.4(a) and the Poisson homomorphism of its part (b). Clearly, it will be the next important step to prove that the assumptions (super-toric and geometrically generic) also apply to other clusters.

We have a similar story, when \mathfrak{g} is a symmetric Kac-Moody Lie algebra, and W its Weyl group. The unipotent radicals U_w , associated to each $w \in W$, together with the standard Poisson structure, admit a stratification (see e.g. [53]) satisfying COS. Indeed the strata are labeled by the elements $v \in W$ such that $v \leq w$. And Geiss, Leclerc and Schröer proved that these algebras have a cluster algebra structure ([13]). We believe, again, that each cluster satisfies the conditions of Theorem 5.4.

Of course, COS is trivially satisfied in the case when \mathcal{A} is an acyclic cluster algebra without coefficients with exchange matrix of full rank (see Section 4). This motivates the following conjecture.

Conjecture 5.8. *Let \mathbb{A} be a Noetherian Poisson (upper) cluster algebra with an exchange matrix of full rank. Then \mathbb{A} satisfies COS.*

Moreover, computations for a number of examples and Conjecture 5.8 suggest the following very strong statement.

Conjecture 5.9. *Let \mathbb{A} be a Noetherian Poisson (upper) cluster algebra with exchange matrix of full rank. Let \mathcal{I} be a TPP and suppose that \mathbf{x} is a defining cluster for \mathcal{I} with $x_i \notin \mathcal{I}$ for some $1 \leq i \leq n$. There exists a TPP \mathcal{J} such that $\mathcal{J} \cap \mathbf{x} = (\mathcal{I} \cap \mathbf{x}) \cup \{x_i\}$ if and only if $P_i \in \mathcal{I}$ or x_i is a coefficient.*

Remark 5.10. *The "only if" direction is of course Theorem 3.2. The "if" part would allow us to compute the stratification simply using the matrix B and its mutations without having to know more about the algebra.*

APPENDIX A. TORIC PRIME IDEALS

In this appendix we review some facts regarding prime ideals stable under the action of some torus, essentially following the discussion by Brown and Goodearl in [3, Ch.II], and similar to [18]. Let H be a group acting by automorphisms on a ring R . An ideal I of R is called H -stable if $h(I) = I$ for all $h \in H$. For convenience we will write H -ideals to denote H -stable ideals. We say that R is H -prime if R is nonzero and the product of two non-zero H -ideals is non-zero. An H -prime ideal of R is any proper H -ideal I such that R/I is an H -prime ring. For any ideal I in R we denote by $(I : H)$ the largest H -ideal containing I , i.e.

$$(I : H) = \bigcap_{h \in H} h(I).$$

Note the following facts.

Lemma A.1. [3][II.1.9] *Let H be a group acting by automorphisms on a ring R . If P is a prime ideal, then $(H:P)$ is an H -prime ideal.*

Lemma A.2. [3][II.1.12] *Let R be a Noetherian ring, and suppose that a k -torus H acts on R by automorphisms. If k is algebraically closed, then all H -prime ideals of R are prime.*

We derive the following corollary.

Lemma A.3. *Let R be a Noetherian ring, and suppose that a k -torus H acts on R by automorphisms and that k is algebraically closed. Let I be an H -ideal and let P be a minimal prime over I . Then P is an H -prime.*

Proof. Suppose not. Then $I \subset (P : H) \subset P$. Lemma A.1 yields that $(P : H)$ is an H -prime ideal, while one derives from Lemma A.2 that $(P : H)$ is a prime ideal. One concludes that $P = (P : H)$ since P was assumed to be minimal. The lemma is proved. \square

Note that we have not required that $(P : H)$ be Poisson. We will call a Poisson structure on a k -algebra A *compatible* with the action of the torus H if H acts by Poisson automorphisms, that means

$$\{h(x), h(y)\} = h(\{x, y\}) .$$

We have the following fact.

Lemma A.4. *Let $(A, \{\cdot, \cdot\})$ be a Poisson algebra and suppose that a k -torus H acts on A compatibly. If I is a Poisson ideal, then $(I : H)$ is a Poisson ideal as well.*

Proof. Notice that $h(I)$ is a Poisson ideal for all $h \in H$. Indeed, we obtain for all $x \in I$ and $y \in A$

$$\{h(x), y\} = \{h(x), h(h^{-1}(y))\} = h(\{x, h^{-1}(y)\}) \in h(I) .$$

We conclude immediately that $(I : H)$ is a Poisson ideal. \square

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